Head-on collision of ultrarelativistic charges

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Dedication: It is a pleasure to dedicate this article to Vincent Moncrief on the occasion of his special birthday.

Abstract

We consider the head-on collision of two opposite-charged point particles moving at the speed of light. Starting from the field of a single charge we derive in a first step the field generated by uniformly accelerated charge in the limit of infinite acceleration. From this we then calculate explicitly the burst of radiation emitted from the head-on collision of two charges and discuss its distributional structure. The motivation for our investigation comes from the corresponding gravitational situation where the head-on collision of two ultrarelativistic particles (black holes) has recently aroused renewed interest.

UWThPh–2002–44
TUW 03–10

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Introduction

Sources moving with high velocity, close to the speed of light, are of general interest in physics, e.g. for studying phenomena in high energy particle physics or for understanding radiation from high speed encounters of black holes. Essential simplification in the description of such phenomena is achieved by boosting the source to the velocity of light. Because only massless sources can travel with the fundamental velocity, this limit requires some care, letting the rest mass tend to zero while at the same time keeping the energy constant. Such light-like sources can provide a leading-order approximation for describing massive sources traveling with ultra-relativistic velocities. In this paper we study classical solutions of Maxwells equations for special light-like currents, i.e. charges moving with the velocity of light in Minkowski space-time. Consider, as the simplest example, a moving point charge. By boosting the charge to higher and higher velocities the originally spherical symmetric Coulomb field becomes, as it is well known, deformed and more and more concentrated in the plane orthogonal to the charges. Finally, in the limit, this deformation becomes extreme, the field becomes completely concentrated on the null-hyperplane which contains the light-like trajectory of the charge. From the mathematical point of view care must be taken when calculating the electromagnetic field associated light-like currents by the Green-function method. The reason being, that the trajectory of light-like sources lie in the characteristic surfaces of the hyperbolic system. Simply speaking, the light cone of the point where the field is to be evaluated may intersect the source at null infinity. These contributions however are essential for obtaining the correct field.

Our main goal here is to present an exact solution of Maxwell's equation which is the classical analog of pair annihilation in quantum field theory: Two opposite charged point sources moving with the speed of light collide head on and produce a burst of electromagnetic radiation. After the collision no charge but only radiation is present. Of course this is an idealised situation but does provide an approximation to what happen in more realistic situations. The description of colliding charges may also provide better insight for understanding the radiation emitted by the collision of two black holes as first obtained by D'Eath [1]. More recently, the gravitational interaction of two colliding massless particles [2] to produce a black hole was discussed [3, 4]. Since the sources and fields are distributional in nature, the electromagnetic setting disentangles the distributional from the geometrical aspects.

1) The EM-pulse

Since the situation we are interested in has null (lightlike) character we adapt our coordinates accordingly, i.e.

$$ds^2 = -2dudv + d\tilde{x}^2,$$

where $\tilde{x}$ refers to coordinates in the orthogonal space to the timelike 2-plane spanned by $l^a = \partial_v^a$ and $n^a = \partial_u^a$ [1].

The current of a point-charge e moving at ultrarelativistic (i.e. the velocity of light) speed along the direction of l^a is given by

$$j = e\delta(u)\delta^{(2)}(\tilde{x})\partial_v. \tag{1}$$

The corresponding electromagnetic field is obtained by solving Maxwell's equations for the potential A^a which conveniently reduce to a wave equation in the Lorenz-gauge ($\partial_a A^a = 0$)

$$\partial^2 A^a = 4\pi j^a.$$

Formally the potential may be obtained by employing the Green-function for the wave operator

$$G(x) = \int \frac{d^4k}{(2\pi)^4} \frac{-1}{k^2} e^{ikx}.$$

Here (and in the following) the Fourier-expressions will be understood in terms of symbolic calculus, i.e. as representing a solution of the equation

$$\partial^2 G(x) = \delta^{(4)}(x).$$

For the Fourier-transform of the current (1)

$$j^a(k) = e2\pi\delta(k^n)l^a \qquad k^a =: k^l l^a + k^n n^a + \tilde{k}^a$$

this yields

$$A^a(x) = 4\pi e \int \frac{d^4k}{(2\pi)^4} e^{ikx} \frac{-1}{k^2} 2\pi\delta(k^n) l^a.$$

Decomposing $k \cdot x$ and k^2 with respect to l^a and n^a

$$k^2 = -2k^l k^n + \tilde{k}^2 \qquad kx = -k^n v - k^l u + \tilde{k}\tilde{x}$$

[1] Here and in the following small latin sub- and superscripts from the beginning of the alphabet will denote abstract indices referring to the tensor type only

this reduces to

$$A^a(x) = 4\pi e l^a \int \frac{d^2\tilde{k}}{(2\pi)^2}\frac{dk^l}{2\pi}\frac{-1}{\tilde{k}^2}e^{-ik^l u + i\tilde{k}\tilde{x}} = 4\pi e\delta(u) l^a \int \frac{d^2\tilde{k}}{(2\pi)^2}\frac{-1}{\tilde{k}^2}e^{-i\tilde{k}\tilde{x}}.$$

The last expression is nothing but the Fourier-transform of the Green-function for the two-dimensional Laplacian, i.e.

$$\tilde{\partial}^2 f(\tilde{x}) = \delta^{(2)}(\tilde{x}) \qquad \tilde{f} = \frac{1}{2\pi}\log\rho, \quad \rho^2 = \tilde{x}^2.$$

Therefore the potential for the EM-pulse becomes

$$A^a(x) = 2e\delta(u)\log\rho l^a$$

from which we readily obtain the field-strength

$$F = dA = -2e\delta(u)\frac{1}{\rho}d\rho \wedge du. \tag{2}$$

The charge produces an impulsive electromagnetic field (EM-pulse). As expected this field is concentrated on the null hypersurface $u = 0$ and is singular along the trajectory of the charge $u = \rho = 0$. In deriving F_{ab} we have implicitly chosen boundary conditions. Actually calculating the field explicitly with retarded or advanced Green-functions leads to the same expression. For timelike currents the retarded (advanced) field implies that there is no incoming (outgoing) radiation. This means that the field falls off faster than $1/r$ at null-infinity. Because in our case the charge comes from past and leaves for future null-infinity, this condition cannot be satisfied. Moreover, as pointed out, F_{ab} as given by (2), can be obtained from an ultrarelativistic boost of the Coulomb field. Since the Coulomb field contains no radiation, we claim that (2) carries only radiation that is necessarily associated to the light-like current (1).

As usual the electric and magnetic parts of F_{ab} may be obtained by projecting the field-strength and its dual onto a timelike direction u^a

$$E = (\partial_t \lrcorner F) = -\frac{\sqrt{2}e}{\rho}\delta(u)d\rho \qquad B = (\partial_t \lrcorner * F) = -\sqrt{2}e\delta(u)d\phi.$$

(Due to the null-character of the system this decomposition is the same for any observer with the same transversal $\tilde{x}$-space) Note that the field has a purely radiative character, i.e. both invariants of F_{ab} vanish, except at the location of the charge.

2) The Hook-current

We proceed with the so-called “hook-current”, i.e. the electromagnetic field generated by a point-charge that moves ultrarelativistically (i.e. the speed of light) and undergoes a sudden change of direction. This current can also be considered to be the limit of the current due to a constantly accelerated point-charge in the infinite acceleration limit. The expression for the hook-current is given by

$$j = j^n n + j^l l = e\delta^{(2)}(x)(\theta(-u)\delta(v)\partial_u + \theta(v)\delta(u)\partial_v)$$

and consists of two parts: j^n resembles a charge e moving in the direction n^a along $v = 0 = \tilde{x}^i$ up to the point $u = v = 0 = \tilde{x}^i$ where its motion is reversed. j^l is the current in the direction l^a along $u = 0 = \tilde{x}^i$. (See Fig.1)

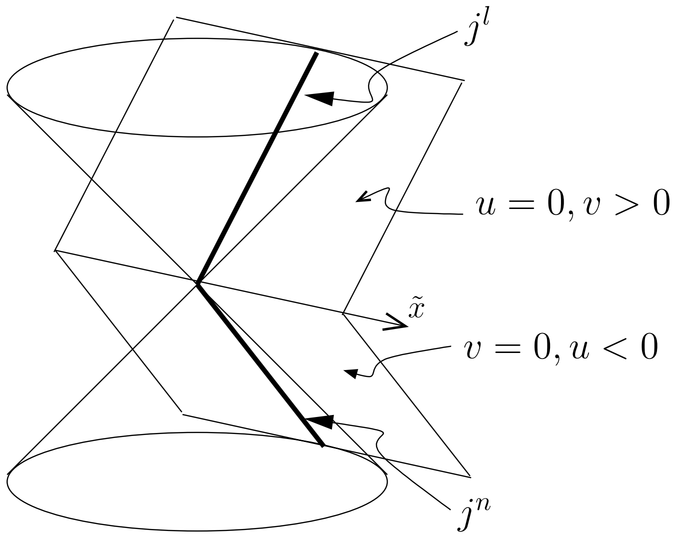

Fig.1 The spacetime diagram for the ”hook-current”

(Charge conservation for the hook-current, i.e. $\partial_a j^a = 0$, can be easily verified)

As in the previous case we solve Maxwell’s equations in the Lorenz gauge and adapt the coordinates to the timelike 2-plane spanned by l^a and n^a. Decomposing the potential

$$A = A^l l + A^n n$$

this boils down to two scalar wave-equations

$$\partial^2 A^l = 4\pi e\delta^{(2)}(x)\delta(u)\theta(v) \qquad \partial^2 A^n = 4\pi e\delta^{(2)}(x)\delta(v)\theta(-u). \tag{3}$$

The solution of these equations is found by a slight detour: Differentiating the A^l-equation with respect to v

$$\partial^2 \partial_v A^l = 4\pi e\delta^{(4)}(x)$$

and using the identity $\partial^2\delta_+(x^2) = -2\pi\delta^{(4)}(x)$ (cf. appendix) immediately gives

$$\partial_v A^l = -2e\delta_+(x^2) \tag{4}$$

where $\delta_+(x^2)$ denotes the delta-function concentrated on the future light-cone of the origin in Minkowski space. More rigorously, it is defined as the functional

$$(\delta_+(x^2), \varphi) := \frac{1}{2}\int_0^\infty du \int_0^\infty dv \int d\phi \varphi(u, v, \sqrt{2uv}e_\rho), \quad e_\rho = (\cos\phi, \sin\phi) \tag{5}$$

which arises from "splitting" the action of $\delta(x^2)$ into its future and past-part. From the above definition we see that we may symbolically write

$$\delta_+(x^2) = \theta(u)\theta(v)\delta(2uv - \rho^2).$$

Using this identity we may readily integrate $\partial_v A^l = -2e\delta_+(x^2)$ obtaining

$$A^l = -e\theta(u)\theta(v)\frac{1}{u}\theta(2uv - \rho^2). \tag{6}$$

At first glance, due to the appearance of the $\theta(u)/u$ factor, which has a non-integrable singularity at $u = 0$, the result does not seem to be well defined on all test functions and some kind of extension has to be used. In the appendix however we show that the r.h.s. of (6) is a well-defined distribution. Moreover we proof that A^l does indeed satisfy (4).

To find A^n we decompose the current on the r.h.s. of (3) by making use of $\theta(-u) = 1 - \theta(u)$:

$$\partial^2 A^n = 4\pi e\delta(v)\delta^{(2)}(\tilde{x}) - 4\pi e\theta(u)\delta(v)\delta^{(2)}(\tilde{x})$$

Now upon exchanging u and v the second term is precisely of the form as those in the A^l-equation. Thus we may follow the same steps and obtain immediately

$$e\theta(v)\theta(u)\frac{1}{v}\theta(2uv - \rho^2).$$

On the other hand the first term on the r.h.s. of the A^n-equation corresponds to a point charge moving at the velocity of light along $v = 0 = \tilde{x}^i$, i.e. in the direction of n^a. In the previous paragraph we have derived the potential for a charge moving in the direction of l^a. Making the corresponding changes and adding both terms we finally obtain for A^n

$$A^n = 2e\delta(v)\log\rho + e\theta(v)\theta(u)\frac{1}{v}\theta(2uv - \rho^2)$$

and total potential for the hook current can be written as

$$A = e\theta(u)\theta(v)\theta(2uv - \rho^2)(\frac{1}{u}du - \frac{1}{v}dv) - 2e\delta(v)\log\rho dv.$$

From this expression it is straightforward to calculate the field-strength F_{ab}, which becomes

$$F = -4e\delta_+(x^2)du \wedge dv - 2e\delta_+(x^2)\rho d\rho \wedge (\frac{1}{u}du - \frac{1}{v}dv) - 2e\delta(v)\frac{1}{\rho}d\rho \wedge dv.$$

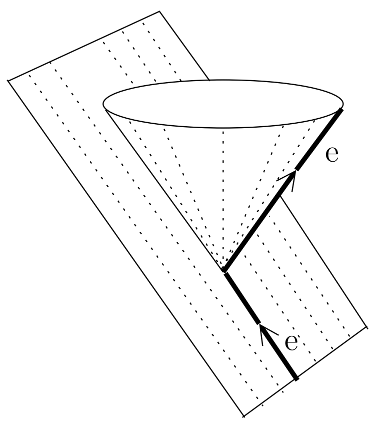

Fig.2 The radiation produced by the "hook-current"

The physical interpretation is as follows: The point-charge comes from past null-infinity. At the instant its motion is reversed, a spherical pulse of radiation is released. This field is concentrated on the forward light-cone and has the properties of a pure radiation field, i.e. both invariants of F_{ab} vanish except on the trajectory of the outgoing charge, where the field diverges. In addition there is the radiation field concentrated on the null hyperplane of the incoming charge. (See Fig.2)

3) Head on collision

From the field of the hook-current we proceed to construct the field generated by a head-on collision of two ultrarelativistic particles with opposite charge. The field of the hook-current can be thought of as the superposition of the field due to the incoming uniformly moving charge +e and the field that is produced by a pair of charges +e and -e . These charges emerge from the vertex of the light-cone and travel with the speed of light together with the spherical pulse, in opposite directions. By superposing the two configurations on the forward light-cone the current of the charge -e cancels exactly the one from +e (which runs in the same direction).

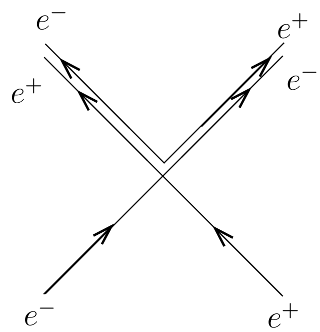

Fig.3 currents of the head-on collision

Now the field for a head-on collision can be constructed as follows: Consider now the field of two charges +e and -e along the directions n^a and l^a respectively

$$F_1 = 2e\delta(u)\frac{1}{\rho}d\rho \wedge du - 2e\delta(v)\frac{1}{\rho}d\rho \wedge dv. \tag{7}$$

Superimpose to the field associated with these charges the field of a pair of charges $-e$, $+e$ starting from the point of collision of the incoming charges (as considered above)

$$F_2 = -2e\delta_+(x^2)\left[2du \wedge dv + \rho d\rho \wedge (\frac{du}{u} - \frac{dv}{v})\right] \tag{8}$$

Thus the total field is the sum of the fields F_1 and F_2. On the forward light-cone with vertex at the collision event, the currents cancel and a pure radiation field with no sources remains.

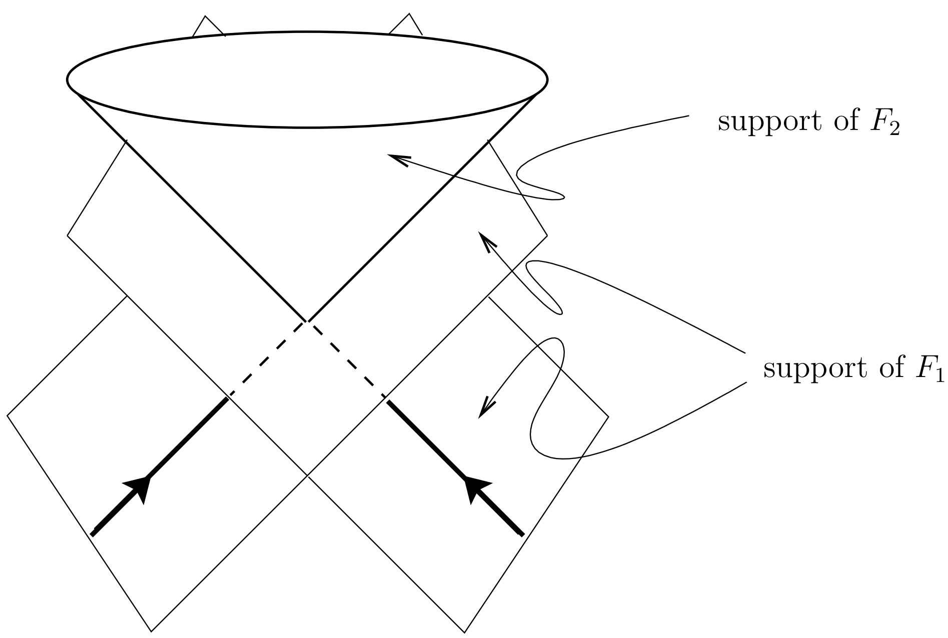

Fig.4 The EM field of two colliding charges. The field consists of the incoming part F_1 plus the radiation F_2 produced $F_1$by the impact

In what follows we discuss the radiation (8) produced by the head-on collision

in more detail. Since

$$F_2 \propto \delta_+(x^2) = \frac{1}{2r}\delta(t-r), \;\; r^2 = \rho^2 + z^2$$

this part is an outgoing spherical pulse centered around the collision point. Its overall strength decreases as $1/r$, but the field is not spherically symmetric. However, since the physical situation is axisymmetric about the spatial direction of the current (z-axis) the field carries this symmetry. Decomposing F_2 into its electric and magnetic parts with respect to ∂_t we find

$$\partial_t \lrcorner F_2 = -\frac{4e}{\rho}\delta_+(x^2)(\rho dz - z d\rho)$$

i.e.

$$E = -\frac{4e}{\rho}\delta_+(x^2)(\rho e_z - z e_\rho)$$

and from

$$*F_2 = -2e\delta_+(x^2)\left[-2d^2\tilde{x} + (\frac{du}{u} + \frac{dv}{v})\rho^2 d\phi\right]$$

and

$$\partial_t \lrcorner * F_2 = -2e\delta_+(x^2)\left[\frac{1}{\sqrt{2}}(\frac{1}{u} + \frac{1}{v})\rho^2 d\phi\right] = -4e\delta_+(x^2) t d\phi$$

i.e.

$$B = -4e\delta_+(x^2)\frac{\sqrt{\rho^2 + z^2}}{\rho} e_\phi$$

where e_ρ, e_ϕ, e_z denote the orthonormal vectors along the corresponding coordinates. In order to show that both Maxwell invariants vanish we "smear" E and B with the same time-dependent test-function $\varphi(t)$, i.e.

$$\begin{aligned} E_\varphi &:= (E, \varphi(t)) = -\frac{4e}{\rho}(\delta_+, \varphi)(\rho e_z - z e_\rho) \\ B_\varphi &:= (B, \varphi(t)) = -\frac{4e}{\rho}(\delta_+, \varphi)\sqrt{\rho^2 + z^2} e_\phi \end{aligned} \tag{9}$$

which are both regular functionals of the spatial variables. It is now easy to see that

$$E_\varphi^2 = 16e^2(\delta_+, \varphi)^2 \frac{\rho^2 + z^2}{\rho^2} = B_\varphi^2 \qquad E_\varphi \cdot B_\varphi = 0$$

for arbitrary smearing functions φ. This holds everywhere except for $\rho = 0$ where there is an additional $1/\rho$ singularity. However, it is precisely along

the two lines on the light cone where F_1 contributes. Adding F_1 shows that the $1/\rho$ singularity cancels and the total field is of the form

$$F \propto \delta_+(x^2) du \wedge dv; \qquad \text{close to } \rho = 0, x^2 = 0, t > 0. \tag{10}$$

This claim can be strengthened in a rigorous manner using smeared field strengths, i.e.

$$\begin{aligned}
F_1(\varphi) &:= (F_1, \varphi(t)) = 2\sqrt{2}e \left(\frac{1}{\rho} d\rho \wedge du \varphi(r\cos\theta) - \frac{1}{\rho} d\rho \wedge dv \varphi(-r\cos\theta) \right) \\
F_2(\varphi) &:= (F_2, \varphi(t)) = -\frac{2e}{r}\varphi(r) \left(du \wedge dv + \right. \\
& \left. \frac{1}{\sqrt{2}\sin\theta} \left[(1+\cos\theta) d\rho \wedge du - (1-\cos\theta) d\rho \wedge dv \right] \right) \\
& \qquad supp(\varphi) = \{ t | t > t_0 \geq 0 \} .
\end{aligned}$$

Adding $F_1(\varphi)$ and $F_2(\varphi)$ toghether and taking the limits $\theta \to 0,\pi$ one immedeately finds

$$\begin{aligned}
F_1(\varphi) &\overset{\theta\to 0}{\sim} \frac{2\sqrt{2}e}{r\sin\theta}\varphi(r) d\rho \wedge du, \\
F_2(\varphi) &\overset{\theta\to 0}{\sim} -\frac{2e}{r}\varphi(r) du \wedge dv - \frac{2\sqrt{2}e}{r\sin\theta}\varphi(r) d\rho \wedge du \\
F_1(\varphi) &\overset{\theta\to \pi}{\sim} -\frac{2\sqrt{2}e}{r\sin\theta}\varphi(r) d\rho \wedge dv, \\
F_2(\varphi) &\overset{\theta\to \pi}{\sim} -\frac{2e}{r}\varphi(r) du \wedge dv + \frac{2\sqrt{2}e}{r\sin\theta}\varphi(r) d\rho \wedge dv
\end{aligned}$$

which proves the qualitative claim (10).

From the construction it is evident that the total field satisfies the source-free Maxwell equations everywhere to the future of the collision.

Conclusion

In this work we have explicitly constructed the field due to a charge moving at ultrarelativistic velocities, which undergoes a sudden change of direction, the so-called hook-current. Upon time-reflection of the outgoing charge one obtains the field of the head-on collision, i.e. the classical analogue of pair annihilation leaving a pure radiation field behind.

From a distributional point of view and due to the null-character of the situation our construction is rather delicate, which manifests itself in the fact that the formal expression for the potential seems to contain a non-integrable

singularity thus not giving rise to functional defined on all of test-function space. However a careful analysis shows that it is nevertheless possible to construct a well-defined distribution from the formal expression which satisfies the required differential equation. Recently the problem of colliding ultrelativistic massless particles and the possible formation of black-holes has attracted some interest. Podolsky et al. [5] proposed the null limit of the C-metric [6] as scenario describing the headon-collision of two ultrarelativistic black holes. However, because of the existence of singular strut between the black holes we believe that this is not the appropriate physical situation. The gravitational analogue of the head-on collision is more realistically considered by Giddings et al.[3]. We do hope that our work may shed some light on the corresponding distributional techniques required in the gravitational context.

Appendix

This appendix is devoted to the distributional aspects of our work, namely the definition of $\delta_+(x^2)$ and the construction of the solution of $\partial_u f = \delta_+(x^2)$. Using the definition of δ_+

$$(\delta_+(x^2), \varphi) := \frac{1}{2} \int_0^\infty du \int_0^\infty dv \int d\phi \varphi(u, v, \sqrt{2uv} e_\rho), \tag{11}$$

and its symbolic form

$$\delta_+(x^2) = \theta(u)\theta(v)\delta(2uv - \rho^2),$$

allows a (formal) integration of $\partial_u f = \delta_+(x^2)$

$$f = \theta(u)\theta(v)\frac{1}{v}\theta(2uv - \rho^2). \tag{12}$$

At first glance, due to the appearance of the $\theta(v)/v$ factor, which has a non-integrable singularity at $v = 0$, the result does not seem to be well defined on all test functions and some kind of extension has to be used. Let us briefly recall how this is done for $\theta(v)/v$. Obviously

$$(\frac{\theta(v)}{v}, \varphi) = \int_0^\infty dv \frac{1}{v}\varphi(v)$$

is only defined on test functions that vanish at $v = 0$. An extension to all of test function space is achieved by mapping φ onto an integrable function with this property, i.e.

$$([\frac{\theta(v)}{v}], \varphi) := \int_0^\infty dv \frac{1}{v}(\varphi(v) - \theta(1 - v)\varphi(0)),$$

which coincides with the naive definition when $\varphi(0) = 0$. The subtraction term depends explicitly on the neighborhood of zero (which we have chosen to have radius one). Let us now turn back to $\theta(u)(\theta(v)/v)\theta(uv - \rho^2)$, i.e.

$$\begin{aligned}
(\theta(u)\theta(v)\frac{1}{v}\theta(2uv - \rho^2), \varphi) &= \int_0^\infty du \int_0^\infty dv \frac{1}{v} \int_0^{\sqrt{2uv}} \rho d\rho d\phi \varphi(u, v, \rho e_\rho) \\
&= \int_0^\infty du \int_0^\infty dv \frac{1}{v} \Phi(u, v) \\
\text{with} \qquad \Phi(u, v) &:= \int_0^{\sqrt{2uv}} \rho d\rho d\phi \varphi(u, v, \rho e_\rho)
\end{aligned}$$

Our previous considerations suggest to take a closer look at the small-v behavior of Φ

$$\begin{aligned}\Phi(u,v) &= \Phi(u,0) + v\partial_v\Phi(u,0) + \mathcal{O}(v^2)\\ &= v(2\pi u\varphi(u,0,0,0)) + \mathcal{O}(v^2),\end{aligned}$$

which tells us that Φ has a zero at $v = 0$ and therefore in the light of the extension-argument

$$(\theta(u)\theta(v)\frac{1}{v}\theta(2uv-\rho^2),\varphi) := \int_0^\infty du\int_0^\infty dv\frac{1}{v}\int_0^{\sqrt{2uv}}\rho d\rho d\phi\varphi(u,v,\rho e_\rho)$$

is well defined on all of test function space. Stated differently $\theta(u)(\theta(v)/v)\theta(uv-\rho^2)$ is a well-defined distribution.

Let us now consider the action of the d'Alembertian ∂^2 on $(\theta(u)/u)\theta(v)\theta(2uv-\rho^2)$. As a warm-up and to get acquainted with the neccessary techniques let us first show that $\theta(u)\theta(v)\delta(2uv-\rho^2)$ is the retarded Green-function of ∂^2, i.e.

$$\begin{aligned}(\partial^2(\theta(u)\theta(v)\delta(2uv-\rho^2)),\varphi) &= (\theta(u)\theta(v)\delta(2uv-\rho^2),\partial^2\varphi)\\ &= \frac{1}{2}\int_0^\infty du\int_0^\infty dv\int_0^{2\pi} d\phi\,\partial^2\varphi(u,v,\sqrt{2uv}e_\rho).\end{aligned}$$

Taking into account the coordinate decomposition of $\partial^2 = -2\partial_u\partial_v + \delta^{ij}\partial_i\partial_j$ we split the last expression into two parts

$$\begin{aligned}u,v-part &= \frac{1}{2}\int_0^\infty du\int_0^\infty dv\int_0^{2\pi} d\phi\,\partial_u\partial_v\varphi(u,v,\sqrt{2uv}e_\rho)\\ &= \frac{1}{2}\int_0^\infty du\int_0^\infty dv\left\{\partial_u\int_0^\infty d\phi\partial_v\varphi - \sqrt{\frac{v}{2u}}\int_0^{2\pi} d\phi e^i_\rho\partial_v\partial_i\varphi\right\}\\ &= \pi\varphi(0) - \frac{1}{2}\int_0^\infty du\int_0^\infty dv\sqrt{\frac{v}{2u}}\left\{\partial_v\int_0^{2\pi} d\phi\, e^i_\rho\partial_i\varphi - \sqrt{\frac{u}{2v}}\int_0^{2\pi} d\phi e^i_\rho e^j_\rho\partial_i\partial_j\varphi\right\}\\ &= \pi\varphi(0) + \frac{1}{4}\int_0^\infty du\int_0^\infty dv\left\{\frac{1}{\sqrt{2uv}}\int_0^{2\pi} d\phi\, e^i_\rho\partial_i\varphi + \int_0^{2\pi} d\phi e^i_\rho e^j_\rho\partial_i\partial_j\varphi\right\}\end{aligned}$$

$$\begin{aligned}i,j-part &= \frac{1}{2}\int_0^\infty du\int_0^\infty dv\int_0^{2\pi} d\phi\,\delta^{ij}\partial_i\partial_j\varphi(u,v,\sqrt{2uv}e_\rho)\\ &= \frac{1}{2}\int_0^\infty du\int_0^\infty dv\int_0^{2\pi} d\phi\,(e^i_\rho e^j_\rho + e^i_\phi e^j_\phi)\partial_i\partial_j\varphi\\ &= \frac{1}{2}\int_0^\infty du\int_0^\infty dv\left\{\int_0^{2\pi} d\phi e^i_\rho e^j_\rho\partial_i\partial_j\varphi + \frac{1}{\sqrt{2uv}}\int_0^{2\pi} d\phi e^i_\phi\partial_\phi\partial_i\varphi\right\}\\ &= \frac{1}{2}\int_0^\infty du\int_0^\infty dv\left\{\int_0^{2\pi} d\phi e^i_\rho e^j_\rho\partial_i\partial_j\varphi + \frac{1}{\sqrt{2uv}}\int_0^{2\pi} d\phi e^i_\rho\partial_i\varphi\right\}\end{aligned}$$

Adding the u,v–part (multiplied with a factor -2) to the i,j–part finally gives

$$(\partial^2(\theta(u)\theta(v)\delta(2uv-\rho^2)),\varphi) = -2\pi\varphi(0) = -2\pi(\delta^{(4)}(x),\varphi)$$

thus proving the desired result. Proceeding in an analoguous manner we will evaluate the d'Alembertian on $(\theta(u)/u)\theta(v)\theta(2uv-\rho^2)$, i.e.

$$\begin{aligned}(\partial^2(\theta(u)\theta(v)\frac{1}{u}\theta(2uv-\rho^2)),\varphi) &= (\theta(u)\theta(v)\frac{1}{u}\theta(2uv-\rho^2),\partial^2\varphi) \\ &= \int_0^\infty du \int_0^\infty dv \frac{1}{u}\int_0^{\sqrt{2uv}} \rho d\rho \int_0^{2\pi} d\phi\, \partial^2\varphi(u,v,\rho e_\rho)\end{aligned}$$

Splitting into u,v- and i,j–part yields

$$\begin{aligned}u,v-part &= \int_0^\infty du \int_0^\infty dv \frac{1}{u}\int_0^{\sqrt{2uv}} \rho d\rho \int_0^{2\pi} d\phi\, \partial_u\partial_v\varphi(u,v,\rho e_\rho) \\ &= \int_0^\infty du \int_0^\infty dv \left\{ \frac{1}{u}\partial_v \int_0^{\sqrt{2uv}} \rho d\rho \int_0^{2\pi} d\phi\, \partial_u\varphi - \int_0^{2\pi} d\phi\, \partial_u\varphi \right\} \\ &= -\int_0^\infty du \int_0^\infty dv \int_0^{2\pi} d\phi\, \partial_u\varphi \\ &= 2\pi \int_0^\infty dv \varphi(0,v,0) + \int_0^\infty du \int_0^\infty dv \frac{2v}{2\sqrt{2uv}} \int_0^{2\pi} d\phi\, e_\rho^i \partial_i\varphi\end{aligned}$$

and

$$\begin{aligned}i,j-part &= \int_0^\infty du \int_0^\infty dv \frac{1}{u}\int_0^{\sqrt{2uv}} \rho d\rho \int_0^{2\pi} d\phi\, \partial^i\partial_i\varphi(u,v,\rho e_\rho) \\ &= \int_0^\infty du \int_0^\infty dv \frac{1}{u}\int_0^{2\pi} d\phi\, \sqrt{2uv} e_\rho^i \partial_i\varphi\end{aligned}$$

Adding u,v–part (multiplies by a factor -2) and i,j–part finally gives

$$(\partial^2(\theta(u)\theta(v)\frac{1}{u}\theta(2uv-\rho^2)),\varphi) = -4\pi \int_0^\infty dv\, \varphi(0,v,0) = -4\pi(\theta(v)\delta(u)\delta^{(2)}(x),\varphi),$$

which once again is the desired result.

References

[1] D'Eath P. in *L.Smarr (ed.) Sources of Gravitaional Radiation*, Cambridge University press (1979) 273

[2] Aichelburg P C and Sexl R U, *Gen.Rel.Grav.* **2** (1971) 303

[3] Eardley D M and Giddings S B, *Phys. Rev.* **D 66** (2002) 44011

[4] Kohlprath E and Veneziano G, *JHEP* **0206** (2002) 057

[5] Podolský J and Griffiths J B, *Gen.Rel.Grav.* **33** (2001) 59

[6] Kinnersley W and Walker M, *Phys.Rev.* **D 2** (1970)